Discussion 14 Worksheet

Spherical coordinates and general changes of variables

Date: 10/20/2021

MATH 53 Multivariable Calculus

1 Spherical coordinates

1. Solve for ρ, ϕ, θ in terms of x, y, z. That is, find the inverse of the spherical coordinate mapping. Warning: you may need casework.

2. Describe the following surfaces (defined by Cartesian coordinates) in terms of spherical coordinates).

   $x = \sqrt{3}y$.

   $z^2 = x^2 + y^2$.

   $x^2 + y^2 + z^2/4 = 1$.

3. Find the volume of the region bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = 1$.

4. Compute the following integral over the region R lying above the cone $z^2 = x^2 + y^2$ and below the unit sphere

$$\iiint_R z^2 \, dV$$

5. Let d be a real number and consider the improper integral

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{(x^2 + y^2 + z^2)^d}.$$

   For which values of d does this integral converge? Compute the integral for the values of d that make it converge.
   Hint: As a first step, check that the region of integration is a sphere.

2 Calculating the Jacobian

Find the absolute value of the Jacobian determinant for each of the following changes of coordinates.

1. $x = au + bv$ and $y = cu + dv$.

2. $x = u^2 - v^2$ and $y = 2uv$.

3. $x = e^u \cos(v)$ and $y = e^u \sin(v)$.

4. $x = \frac{u}{u^2+v^2}$ and $y = \frac{-v}{u^2+v^2}$. Note that this transformation is its own inverse, in the sense that we can solve $u = \frac{x}{x^2+y^2}$ and $v = \frac{-y}{x^2+y^2}$. Also check that $(x^2 + y^2)(u^2 + v^2) = 1$.

3 Integrating with change of variables

1. Consider the region $\mathcal{R}$ in the plane: $3x^2 + 4xy + 3y^2 \leq 1$.

   Describe the transformed region using the change of variables $x = v - u$ and $y = u + v$.

   Find the area of $\mathcal{R}$.

2. Let D be the annulus $1 \leq x^2 + y^2 \leq 4$ and consider the integral

$$\iint_D \frac{1}{(x^2+y^2)^2} e^{\frac{x}{x^2+y^2}} \, dxdy.$$

   Perform the change of variables $x = \frac{u}{u^2+v^2}, y = \frac{-v}{u^2+v^2}$ to simplify the integral, but do not evaluate.

4 True/False

Supply convincing reasoning for your answer.

(a) T F If the Jacobian of a transformation $x = x(u,v), y = y(u,v)$ is always non-zero, then the transformation is one-to-one.

(b) T F The image of a rectangle in the plane under the transformation $x = 2u, y = -2v$ will be another rectangle.

(c) T F There is a point with spherical coordinates $\rho = 1/2, \phi = 3\pi/2, \theta = \pi/2$.

(d) T F The "ρ" in spherical coordinates equals the "r" in cylindrical coordinates.

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.